The Complete Guide to Quadratic Curve Sketching

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Contents

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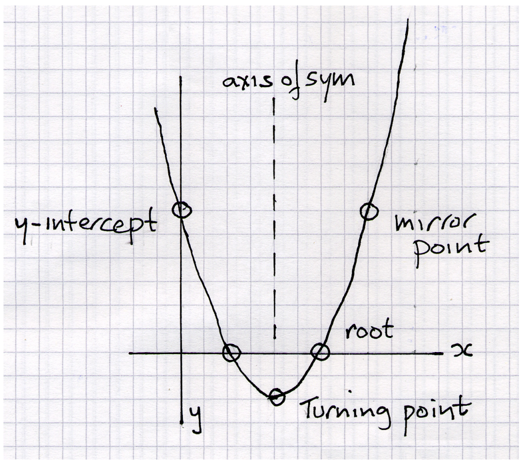
1 Sketching Quadratics

A quadratic is give by the equation

$$y = ax^2 + bx + c$$

There are a number of points and lines that can be found in order to sketch this curve.

- Turning point
- Axis of Symmetry
- Mirror point
- Y intercept
- X intercepts – the real roots



The turning point is always required, and another two points are needed for a rough sketch. The axis of symmetry is useful for laying out the curve. The y-intercept is easy to calculate.

2 Completing the Square

Completing the square is used for transforming $y = ax^2 + bx + c$ into the more helpful form $a(x + h)^2 + k$.

$$ax^2 + bx + c = a(x+h)^2 + k \quad \text{where} \quad h = \frac{b}{2a} \quad \text{and} \quad k = c - \frac{b^2}{4a}$$

Alternatively, the square can be calculated as the following example shows

2.1 Example of Completing the Square

$$y = x^2 + 10x + 28 \tag{1}$$

Take half the x coefficient, and proceed as follows to complete the square

$$(x+5)^2 = x^2 + 10x + 25 \quad \Rightarrow \quad x^2 + 10x = (x+5)^2 - 25$$

So substituting in (1), we get

$$y = x^2 + 10x + 28 = (x+5)^2 - 25 + 28$$

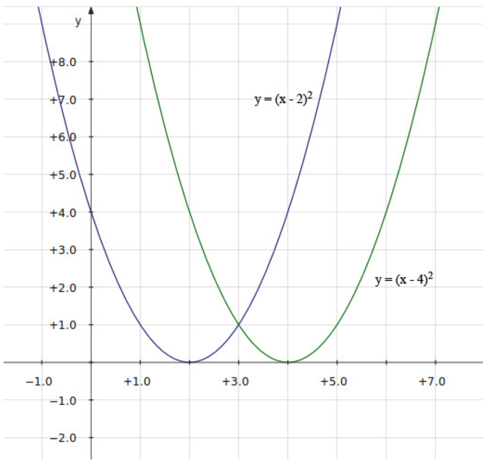
$$y = (x+5)^2 + 3$$

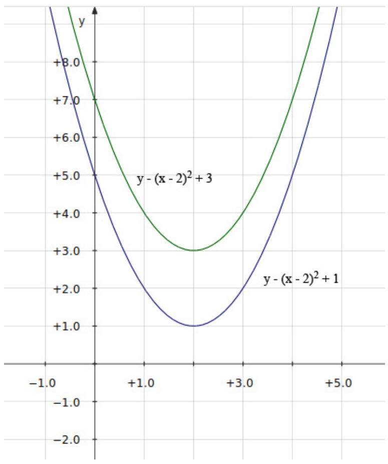
3 Finding the Turning Point

After completing the square we have the form

$$y = a(x+h)^2 + k$$

The h and k values of this equation give the turning point of the quadratic as $(-h, k)$.





When $k = 0$ the quadratics is a 'perfect square,' and has only one real root.

4 Finding the Roots

If the quadratic is in the form

$$y = a(x+h)^2 + k$$

We can easily deduce the number of roots:

$$\begin{array}{lcl} k > 0 & \Rightarrow & \text{no real roots} \\ k = 0 & \Rightarrow & \text{one real roots} \\ k < 0 & \Rightarrow & \text{two real roots} \end{array}$$

Calculating the roots is simple. When $y = 0$

$$a(x+h)^2 + k = 0 \quad \Rightarrow \quad x = \pm\sqrt{-k/a} - h$$

4.1 Example of Calculating Roots

$$y = (x+1)^2 - 16$$

The roots are calculated as follows

$$\begin{aligned} (x+1)^2 - 16 &= 0 \\ (x+1)^2 &= 16 \\ x &= \pm\sqrt{16} - 1 \\ &= -1 \pm 4 \\ &= -5 \text{ or } 3 \end{aligned}$$

5 Sketch Examples

5.1 Example One

$$y = x^2 - 4x$$

Quadratics with no constants coefficient always pass through the origin and have two real roots.

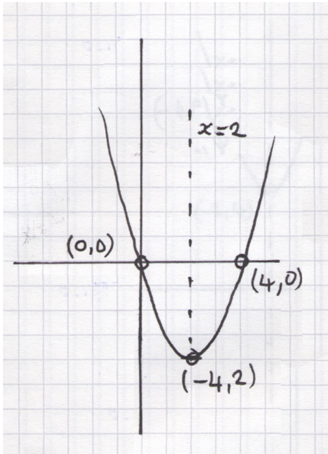
Y intercept: when $x = 0$, $y = 0$ and the point is $(0, 0)$

X intercepts: when $y = 0$, $x^2 - 4x = 0$

$x(x-4) = 0 \Rightarrow x = 0$ or $x = 4$, and the points are $(0, 0)$ and $(4, 0)$

Axis of symmetry is $x = 2$

Turning point: $y = 2^2 - 4 \times 2 = 4 - 8 = -4$, and the point is $(2, -4)$



5.2 Example Two

$$y = x^2 + 6x + 9$$

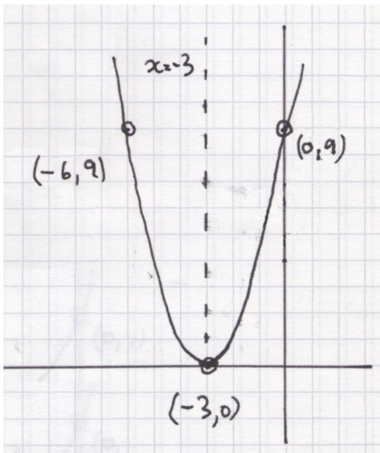
Y intercept: when $x = 0$, $y = 9$ and the point is $(0, 9)$

Complete the square: $(x + 3)^2 = x^2 + 6x + 9$, so it is a perfect square.

$y = (x + 3)^2 \Rightarrow$ one real root and turning point at $(-3, 0)$

Axis of symmetry is $x = -3$

Mirror point is $(-6, 9)$



5.3 Example Three

$$y = x^2 - 4x + 3 \tag{2}$$

Y intercept: when $x = 0$, $y = 3$ and the point is $(0, 3)$

Complete the square: The x coefficient is 4 so

$$(x-2)^2 = x^2 - 4x + 4 \quad \Rightarrow \quad x^2 - 4x = (x-2)^2 - 4$$

Substituting this in (2) gives $y = (x-2)^2 - 4 + 3$, so

$$y = (x-2)^2 - 1$$

Turning Point is $(2, -1) \Rightarrow$ two real roots

Axis of symmetry is $x = 2$

Mirror point is $(4, 3)$

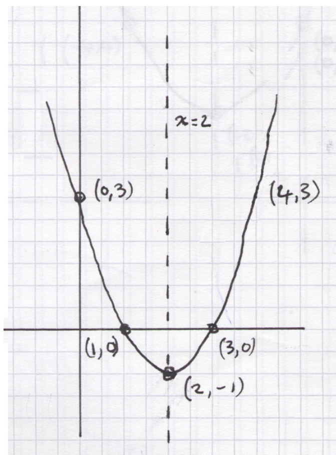
X intercepts: when $y = 0$,

$(x-2)^2 - 1 = 0$

$\Rightarrow (x-2)^2 = 1$

$\Rightarrow x - 2 = \pm 1$

So $x = 1$ or $x = 3$, and the points are $(1, 0)$ and $(3, 0)$



5.4 Example Four

$$x^2 + 4x + 8 \tag{3}$$

Y intercept: when $x = 0$, $y = 8$ and the point is $(0, 8)$

Complete the square:

$$(x + 2)^2 = x^2 + 4x + 4 \quad \Rightarrow \quad (x - 2)^2 - 4 = x^2 + 4x$$

Substituting this in (3) gives $y = (x + 2)^2 - 4 + 8$, so

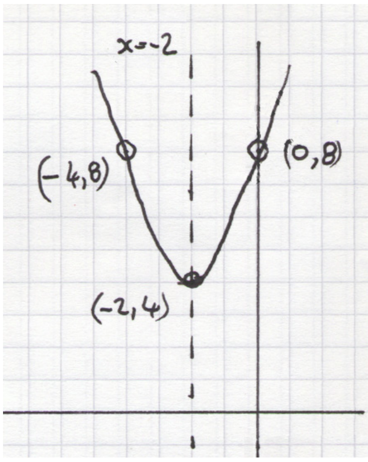
$$y = (x + 2)^2 + 4$$

Turning Point is $(-2, 4) \Rightarrow$ no real roots

Axis of symmetry is $x = -2$

Mirror point is $(-4, 8)$

X intercepts: There are none



5.5 Example Five

$$y = \frac{x^2}{2} + 2x + 4$$

This is an example of a non-monic quadratic. Rearranging it gives

$$y = \frac{x^2 + 4x}{2} + 4 \tag{4}$$

Y intercept: when $x = 0$, $y = 4$ and the point is $(0, 4)$

Complete the square: We do this for $x^2 + 4x$

$$(x+2)^2 = x^2 + 4x + 4 \quad \Rightarrow \quad x^2 + 4x = (x-2)^2 - 4$$

Substituting this in (4) gives

$$y = \frac{(x-2)^2 - 4}{2} + 4 \quad \Rightarrow \quad y = \frac{(x-2)^2}{2} - 2 + 4$$

So

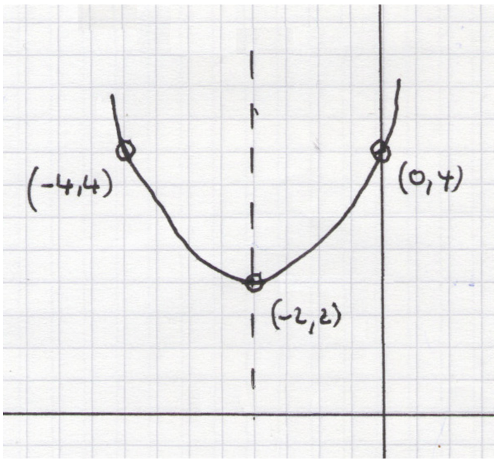
$$y = \frac{(x-2)^2}{2} + 2$$

Turning Point is $(-2, 2) \Rightarrow$ no real roots

Axis of symmetry is $x = -2$

Mirror point is $(-4, 4)$

X intercepts: There are none



6 Further Information

6.1 Completing the Square

$$y = ax^2 + bx + c \quad \rightarrow \quad a(x+h)^2 + k$$

The basic method is applied to quadratics that have $a = 1$, which are called 'monic' quadratics. The method is extended to handle the more general 'non-monic' quadratic where $a \neq 1$.

6.1.1 Monic Quadratics

These are equations that have an x^2 coefficient of one, and thus have the form $y = x^2 + bx + c$.

Consider the 'perfect square'

$$(x+p)^2 = x^2 + 2p + p^2$$

A quadratic is a perfect square if the square of half of its x coefficient equals its constant coefficient. That is $b^2/4 = c$.

Now if we express the square as

$$\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}$$

We can rearrange it as

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4}$$

Introducing the constant coefficient c gives

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}$$

This can be expressed as

$$x^2 + bx + c = (x+h)^2 + k \quad \text{where} \quad h = \frac{b}{2} \quad \text{and} \quad k = c - \frac{b^2}{4}$$

6.1.2 Non-Monic Quadratics

Non-Monic quadratics are those that have an x^2 coefficient that is not one, thus

$$y = ax^2 + bx + c$$

This can be expressed as

$$y = a\left[x^2 + \frac{b}{a}x\right] + c \tag{5}$$

Completing the square for $x^2 + bx/a$

$$\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{bx}{a}x + \frac{b^2}{4a^2} \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = x^2 + \frac{b}{a}x$$

Substituting in (5) above

$$\begin{aligned} y &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \end{aligned}$$

This can be expressed as

$$ax^2 + bx + c = a(x+h)^2 + k \quad \text{where} \quad h = \frac{b}{2a} \quad \text{and} \quad k = c - \frac{b^2}{4a}$$

Note that if $a = 1$ these are the same as above.

6.2 The Standard Roots Equation

The roots of $y = ax^2 + bx + c$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The value $b^2 - 4ac$ is called the determinant, and it defines the nature of the quadratic's roots:

$$\begin{array}{llll} b^2 - 4ac < 0 & \text{or} & b^2 < 4ac & \Rightarrow \text{No real roots} \\ b^2 - 4ac = 0 & \text{or} & b^2 = 4ac & \Rightarrow \text{One real root} \\ b^2 - 4ac > 0 & \text{or} & b^2 > 4ac & \Rightarrow \text{Two real roots} \end{array}$$